Lie systems and integrability conditions of differential equations and some of its applications

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Abstract

The geometric theory of Lie systems is used to establish integrability conditions for several systems of differential equations, in particular some Riccati equations and Ermakov systems. Many different integrability criteria in the literature will be analysed from this new perspective, and some applications in physics will be given.

1   Introduction

Non-autonomous systems of first-order and second-order differential equations appear in many places in physics. For instance, Hamilton equations are systems of first-order differential equations, while Euler–Lagrange equations for regular Lagrangians are systems of second-order differential equations. A system of second-order differential equations in n variables of the form $\ddot{x}^i = F^i(x, \dot{x}, t)$, with $i = 1, \ldots, n$, is related with a system of first-order equations in $2n$ variables:

$$\left\{ \begin{array}{ccl} \dot{x}^i & = & v^i \\ \dot{v}^i & = & F^i(x, v, t) \end{array} \right. , \qquad i = 1, \ldots n \,. \tag{1}$$

Therefore, it is enough to restrict ourselves to study systems of first-order differential equations. From the geometric viewpoint a system

$$\dot{x}^i = X^i(x, t), \qquad i = 1, \ldots, n \,, \tag{2}$$

is associated with the t-dependent vector field $X = X^i(x, t) \, \partial/\partial x^i$ whose integral curves are determined by the solutions of the system.

Unfortunately, there is no general method for solving such equations. Relevant questions about integrability are how to find a particular solution (determined by $x(0) = x_0$), or a r-parameter family of solutions, or even the general solution (a n-parameter family of solutions). Finally, when is it possible to find and how to determine a superposition rule for solutions?

We shall understand that to find a solution means to reduce the problem to carry out some quadratures. For instance, the general solution of the inhomogeneous linear differential equation $dx/dt = b_0(t) + b_1(t)x$ can be found with two quadratures and it is given by

$$x(t) = \exp\left(\int_0^t b_1(s)\,ds\right) \times \left(x_0 + \int_0^t b_0(t')\exp\left(-\int_0^{t'} b_1(s)\,ds\right)dt'\right).$$

Actually, when the systems we are dealing with are linear, there is a *linear superposition principle* allowing us to find the general solution as a linear combination of n particular solutions. For instance, for the harmonic oscillator with a t-dependent angular frequency $\omega(t)$:

$$\ddot{x} = -\omega^2(t)\,x \Longleftrightarrow \begin{cases} \dot{x} &=& v \\ \dot{v} &=& -\omega^2(t)\,x \end{cases},$$

whose solutions are the integral curves of the t-dependent vector field $X = v\partial/\partial x - \omega^2(t)x\,\partial/\partial v$, if we know a particular solution, the general solution can be found by means of one quadrature and if we know two particular solutions, x_1 and x_2, the general solution is a linear combination (no quadrature is needed) $x(t) = k_1\,x_1(t) + k_2\,x_2(t)$.

There are systems whose general solution can be written as a nonlinear function of some particular solutions. For instance, for Riccati equation: if a particular solution is known, the general solution is obtained by two quadratures, if two particular solutions are known the problem reduces to one quadrature and, finally, when three particular solutions are known, x_1, x_2 and x_3, the general solution can be found from the cross-ratio relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k,$$

which provides us a nonlinear superposition rule[1].

There also exist cases in which we can superpose solutions of one system for finding solutions of another one. We have seen one example: the general solution of the inhomogeneous linear equation can be written as $x(t) = x_1(t) + C\,x_0(t)$, where $x_0(t)$ is a solution of the associated homogeneous equation and $x_1(t)$ is a particular solution of the inhomogeneous linear one.

Milne-Pinney equation[2] $\ddot{x} = -\omega^2(t)x + k/x^3$ is usually studied together with the time-dependent harmonic oscillator $\ddot{y} + \omega^2(t)y = 0$ and the system is called Ermakov system. Pinney showed in a short paper [2] that the general solution of the first equation can be written as a nonlinear superposition of two solutions of the associated harmonic oscillator. All these properties can be better understood in the framework of Lie systems, conveniently extended in some cases to include systems of second-order differential equations. These systems have a lot of applications not only in mathematics but also in many different branches of classical and quantum physics.

Let us look for systems admitting a (maybe nonlinear) superposition rule. The main result was given by Lie [3, 4]:

Theorem: *Given (2) a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that the general solution is $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)$, with $\{x_{(a)} \mid a = 1, \ldots, m\}$ being a set of*

particular solutions of the system and $k_1, \ldots, k_n$, are n arbitrary constants, is that the system can be written as

$$\frac{dx^i}{dt} = Z^1(t)\xi_1^i(x) + \cdots + Z^r(t)\xi_r^i(x),$$

where $Z^1, \ldots, Z^r$, are r functions depending only on t and ξ_α^i, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by $X_\alpha \equiv \sum_{i=1}^{n} \xi_\alpha^i(x^1, \ldots, x^n)\partial/\partial x^i$, $\alpha = 1, \ldots, r$, close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that $[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma$. The number r satisfies $r \le mn$.

The condition in the Theorem is that $X(x, t)$ can be written as $X(x, t) = \sum_{\alpha=1}^{r} Z^\alpha(t) X_\alpha(x)$, with X_α as mentioned above.

Non-autonomous systems corresponding to such t-dependent vector fields will be called Lie systems. One instance of Lie system is the Riccati equation[1]

$$\frac{dx(t)}{dt} = b_2(t)\, x^2(t) + b_1(t)\, x(t) + b_0(t) \,, \tag{3}$$

for which $m = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k \implies x = \frac{k\, x_1(x_3 - x_2) + x_2(x_1 - x_3)}{k\, (x_3 - x_2) + (x_1 - x_3)} \,.$$

The associated Lie algebra is generated by X_0, X_1 and X_2 given by

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x\frac{\partial}{\partial x}, \quad X_2 = x^2\frac{\partial}{\partial x},$$

which close on a $\mathfrak{sl}(2, \mathbb{R})$ 3-dimensional real Lie algebra, because

$$[X_0, X_1] = X_0, \quad [X_0, X_2] = 2X_1, \quad [X_1, X_2] = X_2. \tag{4}$$

The time-dependent harmonic oscillator is also an example of physical relevance. It is described by a Hamiltonian

$$H = \frac{1}{2}\frac{p^2}{m(t)} + \frac{1}{2}\, m(t)\omega^2(t)x^2 \,,$$

which gives rise to the dynamics defined by the t-dependent vector field

$$X(x, p, t) = \frac{1}{m(t)}\, p\frac{\partial}{\partial x} - m(t)\omega^2(t)\, x\frac{\partial}{\partial p} \,.$$

If we consider the set of vector fields

$$X_0 = p\frac{\partial}{\partial x}, \quad X_1 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}\right), \quad X_2 = -x\frac{\partial}{\partial p},$$

which close on a $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra with the same commutation relations as (4), the corresponding t-dependent vector field X can be written as a linear combination $X(\cdot,t) = m(t)\omega^2(t)\,X_2(\cdot) + (1/m(t))\,X_0(\cdot)$, i.e. it is a linear combination with t-dependent coefficients $X(\cdot,t) = \sum_{\alpha=0}^{2} b_\alpha(t)X_\alpha(\cdot)$ with $b_0(t) = 1/m(t)$, $b_1(t) = 0$ and $b_2(t) = m(t)\omega^2(t)$.

The prototype of Lie system is the time-dependent right-invariant vector fields in a Lie group G. Let $\{a_1,\ldots,a_r\}$ denote a basis of T_eG. A right-invariant vector field X^R is one such that $X^R(g) = R_{g*e}X^R(e)$. Define X_α^R by $X_\alpha^R(g) = R_{g*e}a_\alpha$. The t–dependent right-invariant vector field

$$\bar{X}(g,t) = -\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha^R(g) \;,$$

defines a Lie system in G whose integral curves are solutions of the system $\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t)\,X_\alpha^R(g)$, and when applying $R_{g^{-1}}$ to both sides we see that $g(t)$ satisfies

$$R_{g^{-1}(t)*g(t)}\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha \in T_eG \;. \tag{5}$$

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. Then, G can be seen as a principal bundle over G/H: $(G,\tau,G/H)$. The X_α^R are τ-projectable on the corresponding fundamental vector fields of the left-action $\lambda : (g,g'H) \in G \times M \to (gg'H) \in M$ given by $-X_\alpha = -X_{a_\alpha}$ with $\tau_{*g}X_\alpha^R(g) = -X_\alpha(gH)$, the projected vector field in M will be $X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha(x)$, and its integral curves are the solutions of the system of differential equations: $\dot{x} = \sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha(x)$. The solution of this last system starting from x_0 is $x(t) = \lambda(g(t),x_0)$, with $g(t)$ being the solution of (5) such that $g(0) = e$. This means that solving such Lie system in G we are simultaneously solving the corresponding problems in all its homogeneous spaces.

2 SODE Lie systems

A system of second order differential equations can be studied through the corresponding system of first-order differential equations as indicated in (1). We call SODE Lie systems those for which the associated first-order one is a Lie system, i.e. it can be written as a linear combination with t-dependent coefficients of vector fields closing on a finite-dimensional real Lie algebra. An example is the 1-dimensional harmonic oscillator with time-dependent frequency, but the same is true for the 2-dimensional isotropic harmonic oscillator with time-dependent frequency, with an associated vector field

$$X = v_1\frac{\partial}{\partial x_1} - \omega^2(t)x_1\,\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial x_2} - \omega^2(t)x_2\,\frac{\partial}{\partial v_2} \;,$$

which is a linear combination $X = X_2 - \omega^2(t)X_1$ with

$$X_1 = x_1\frac{\partial}{\partial v_1} + x_2\frac{\partial}{\partial v_2}\;, \qquad X_2 = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\;,$$

4

and then they close once again on a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$:

$$[X_1, X_2] = 2\,X_3\,, \quad [X_1, X_3] = -X_1\,, \quad [X_2, X_3] = X_2\,, \tag{6}$$

with

$$X_3 = \frac{1}{2}\left(x_1\frac{\partial}{\partial x_1} - v_1\frac{\partial}{\partial v_1} + x_2\frac{\partial}{\partial x_2} - v_2\frac{\partial}{\partial v_2}\right)\,.$$

The search for the superposition rule for a Lie system consists on looking for enough number of first integrals, independent of the time-dependent coefficients, in an extended space in which we consider several replicas of the given vector field.

The 2-dimensional case admits an invariant F given by the first integral $F(x_1, x_2, v_1, v_2) = x_1 v_2 - x_2 v_1$, which can be seen as a partial superposition rule[5]. Actually, if $x_1(t)$ is a solution of the first equation, then we obtain for each real number k the first-order differential equation for the variable x_2, $x_1(t)\,dx_2/dt = k + \dot{x}_1(t)x_2$, from where x_2 can be found to be given by $x_2(t) = k'x_1(t) + k\,x_1(t)\int^t x_1^{-2}(\zeta)\,d\zeta$. With three copies of the same harmonic oscillator, i.e. X_1 and X_2 given by

$$X_1 = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v\frac{\partial}{\partial x}\,, \qquad X_2 = x_1\frac{\partial}{\partial v_1} + x_2\frac{\partial}{\partial v_2} + x\frac{\partial}{\partial v}\,,$$

there exist two independent first integrals $F_1(x_1, x_2, x, v_1, v_2, v) = xv_1 - x_1v$ and $F_2(x_1, x_2, x, v_1, v_2, v) = xv_2 - x_2v$, from where we obtain the expected superposition rule:

$$x = k_1\,x_1 + k_2\,x_2\,, \qquad v = k_1\,v_1 + k_2\,v_2\,.$$

Another interesting non-linear example is the Pinney equation, the second order non-linear differential equation:

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}\,,$$

where k is a constant, with associated t-dependent vector field

$$X = v\frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v}\,,$$

which is a Lie system because it can be written as $X = L_2 - \omega^2(t)L_1$, where $L_1 := x\,\partial/\partial v$ and $L_2 = (k/x^3)\,\partial/\partial v + v\,\partial/\partial x$ generate a three-dimensional real Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ with nonzero defining relations similar to (6) with $L_3 = (1/2)\,(x\,\partial/\partial x - v\,\partial/\partial v)$.

Note that this isotonic oscillator shares with the harmonic one the property of having a period independent of the energy, i.e. they are isochronous, and in the quantum case they have a equispaced spectrum. The fact that they have the same associated Lie algebra means that they can be solved simultaneously in the group $SL(2, \mathbb{R})$ by the same equation

$$R_{g^{-1}*g}\dot{g} = \omega^2(t)\,a_1 - a_2\,, \quad g(0) = e\,.$$

3 Ermakov systems

We can consider the generalised Ermakov system given by:

$$
\begin{cases}
\ddot{x} & = & \dfrac{1}{x^3}f(y/x) - \omega^2(t)x \\
\ddot{y} & = & \dfrac{1}{y^3}g(y/x) - \omega^2(t)y
\end{cases}
$$

which when $f(u) = k$ and $g(u) = 0$ reduces to the Ermakov system.

This system is described by the t-dependent vector field

$$
X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial v_y} + \left(-\omega^2(t)x + \frac{1}{x^3}f(y/x)\right)\frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}g(y/x)\right)\frac{\partial}{\partial v_y},
$$

which can be written as a linear combination $X = N_2 - \omega^2(t)\,N_1$, where N_1 and N_2 are the vector fields

$$
N_1 = x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y}, \quad N_2 = v_x\frac{\partial}{\partial x} + \frac{1}{x^3}f(y/x)\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial y} + \frac{1}{y^3}g(y/x)\frac{\partial}{\partial v_y},
$$

that generate a three-dimensional real Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ with a third generator

$$
N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y}\right).
$$

There exists a first integral for the motion, $F : \mathbb{R}^4 \to \mathbb{R}$, for any $\omega^2(t)$, which satisfies $N_iF = 0$ for $i = 1,\ldots,3$, but as $[N_1, N_2] = 2N_3$ it is enough to impose $N_1F = N_2F = 0$. The condition $N_1F = 0$, implies that there exists a function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$ such that $F(x,y,v_x,v_y) = \bar{F}(x,y,\xi = xv_y - yv_x)$. Then using the method of the the characteristics in condition $N_2F = 0$, we can obtain the first integral:

$$
F(x,y,v_x,v_y) = \frac{1}{2}(xv_y - yv_x)^2 + \int^{x/y}\left[-\frac{1}{u^3}f\left(\frac{1}{u}\right) + u\,g\left(\frac{1}{u}\right)\right]du.
$$

For the Ermakov system with $f(1/u) = k$ and $g(1/u) = 0$ we obtain the known Ermakov invariant

$$
F(x,y,v_x,v_y) = \frac{k}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{2}(xv_y - yv_x)^2
$$

We can now consider a system made up by a Pinney equation with two associated harmonic oscillator equations, with associated t-dependent vector field

$$
X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{y^3}\frac{\partial}{\partial v_y} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right)
$$

which can be expressed as $X = N_2 - \omega^2(t)N_1$ where N_1 and N_2 are:

$$
N_1 = y\frac{\partial}{\partial v_y} + x\frac{\partial}{\partial v_x} + z\frac{\partial}{\partial v_z}, \quad N_2 = v_y\frac{\partial}{\partial y} + \frac{1}{y^3}\frac{\partial}{\partial v_y} + v_x\frac{\partial}{\partial x} + v_z\frac{\partial}{\partial z},
$$

These vector fields generate a three-dimensional real Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ with the vector field N_3 given by

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial z} - v_z\frac{\partial}{\partial v_z}\right) \ .$$

In this case there exist three first integrals for the distribution generated by these fundamental vector fields: The Ermakov invariant I_1 of the subsystem involving variables x and y, the Ermakov invariant I_2 of the subsystem involving variables y and z, and finally, the Wronskian W of the subsystem involving variables x and z. They are given by $W = xv_z - zv_x$,

$$I_1 = \frac{1}{2}\left((yv_x - xv_y)^2 + c\left(\frac{x}{y}\right)^2\right) \ , \quad I_2 = \frac{1}{2}\left((yv_z - zv_y)^2 + c\left(\frac{z}{y}\right)^2\right) \ .$$

In terms of these three integrals we can obtain an explicit expression of y in terms of x, z and the integrals I_1, I_2, W:

$$y = \frac{\sqrt{2}}{W}\left(I_2 x^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - cW^2}xz\right)^{1/2} \ .$$

This can be interpreted as saying that there is a superposition rule allowing us to express the general solution of the Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with time-dependent frequency.

4   The reduction method and integrability criteria

Given an equation (5) on a Lie group, it may happen that the only non-vanishing coefficients are those corresponding to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and then the equation reduces to a simpler equation on a subgroup, involving less coordinates. An important result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the isotopy subgroup.

One can show that there is an action of the group $\mathcal{G}$ of curves in G on the set of right-invariant Lie systems in G (see e.g. [6] for a geometric justification), and we can take advantage of such an action for transforming a given Lie system into another simpler one.

So, if $g(t)$ is a solution of the given Lie system and we choose a curve $g'(t)$ in the group G, and define a curve $\overline{g}(t)$ by $\overline{g}(t) = g'(t)g(t)$, then the new curve in G, $\overline{g}(t)$, determines a new Lie system. Indeed,

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}\left(g'(t)\right)\mathrm{a}_\alpha \ ,$$

which is similar to the original one, with a different right-hand side. Therefore, the aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right hand side lies in $T_e H$, and hence $\overline{g}(t) \in H$ for all t. This can

be done when we know a solution of the associated Lie system in G/H allows us to reduce the problem to one in the subgroup H[7].

Theorem: *Each solution of (5) on the group G can be written in the form $g(t) = g_1(t)\, h(t)$, where $g_1(t)$ is a curve on G projecting onto a solution $\tilde{g}_1(t)$ for the left action λ of G on the homogeneous space G/H and $h(t)$ is a solution of an equation but for the subgroup H, given explicitly by*

$$(R_{h^{-1}*h}\dot{h})(t) = -\mathrm{Ad}\,(g_1^{-1}(t))\left(\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha + (R_{g_1^{-1}*g_1}\dot{g}_1)(t)\right) \in T_e H\ .$$

This fact is very important because one can show that Lie systems associated with solvable Lie algebras are solvable by quadratures and therefore, given a Lie system with an arbitrary G having a solvable subgroup, we should look for a possible transformation from the original system to one which reduces to the subalgebra and therefore integrable by quadratures.

By the last Theorem there always exists a curve in G that transforms the initial Lie system into a new one related with solvable a Lie subgroup of G. Nevertheless, it can be difficult to find out a solution of the equation in M that determines this transformation. Then, to be able to obtain one is more interesting to suppose also that this transformation is a curve in a certain subset of G, i.e. a one-dimensional subgrop. It would be easier to obtain a transformation but it may be that such a transfromation does not exist. In summary: *The conditions for the existence of such a transformation of a certain form are integrability conditions for the system.*

We could choose for showing this assertion a particular example: Riccati equation. One can find in the literature a lot of integrability criteria for Riccati equation [8, 9, 10], all of them particular examples of the above method [11]. We can also consider other equivalent examples as the Pinney equation, the (generalized) Ermakov system or more relevant examples in Physics, for instance, time-dependent harmonic oscillators. The results obtained for one system are valid for the other; they are essentially conditions for the equation in the group, and all are examples of Lie systems associated with the same Lie group: $SL(2,\mathbb{R})$. Consider, for instance, the Riccati equation (3). The group $SL(2,\mathbb{R})$ contains the affine group (either the one generated by X_0 and X_1 or the one generated by X_1 and X_2), which is SOLVABLE. Therefore, a transformation from the given equation to one of this subgroup allows us to express the general solution in terms of quadratures. This happens when we know a particular solution x_1 of the given equation: $x = x_1 + z$, what corresponds to choose

$$\bar{g}(t) = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}$$

reduces the equation to $dz/dt = (2\,b_2\,x_1 + b_1)z + b_2\,z^2$. The reduction by the knowledge of two or three quadratures has also been studied from this perspective and similarly for the Strelchenya criterion [10].

Each Riccati equation can be considered as a curve in $\mathbb{R}^3$ and we can transform every function in $\mathbb{R}$, $x(t)$, under an element of the group $\mathcal{G}$ of smooth

$SL(2, \mathbb{R})$-valued curves $\mathrm{Map}(\mathbb{R}, SL(2, \mathbb{R}))$, as follows:

$$
\begin{aligned}
\Theta(A, x(t)) &= \frac{\alpha(t)x(t)+\beta(t)}{\gamma(t)x(t)+\delta(t)}, \quad \text{if } x(t) \neq -\frac{\delta(t)}{\gamma(t)}, \\
\Theta(A, \infty) &= \alpha(t)/\gamma(t), \quad \Theta(A, -\delta(t)/\gamma(t)) = \infty, \\
&\text{when } A = \left(\begin{array}{cc} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{array} \right) \in \mathcal{G}.
\end{aligned}
$$

The image $x'(t) = \Theta(\bar{A}(t), x(t))$ of a curve $x(t)$ solution of the given Riccati equation satisfies a new Riccati equation with the coefficients b_2', b_1', b_0':

$$
\begin{aligned}
b_2' &= \delta^2 b_2 - \delta\gamma b_1 + \gamma^2 b_0 + \gamma\dot{\delta} - \delta\dot{\gamma}, \\
b_1' &= -2\beta\delta b_2 + (\alpha\delta + \beta\gamma) b_1 - 2\alpha\gamma b_0 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta}, \\
b_0' &= \beta^2 b_2 - \alpha\beta b_1 + \alpha^2 b_0 + \alpha\dot{\beta} - \beta\dot{\alpha}.
\end{aligned}
$$

This expression defines an affine action of the group $\mathcal{G}$ on the set of Riccati equations or analogous Lie systems.

Lie systems in $SL(2, \mathbb{R})$ defined by a constant curve, $\mathrm{a}(t) = \sum_{\alpha=0}^{2} c_\alpha \mathrm{a}_\alpha$, are integrable and the same happens for curves of the form $\mathrm{a}(t) = D(t)\left(\sum_{\alpha=0}^{2} c_\alpha \mathrm{a}_\alpha\right)$, where D is an arbitrary function, because a time reparametrisation reduces the problem to the previous one, i.e. the system is essentially a Lie system on a one-dimensional Lie group.

We can prove the following theorem which is valid for both Riccati equation and any other Lie system with Lie group $SL(2, \mathbb{R})$:

Theorem: *The necessary and sufficient conditions for the existence of a transformation: $y' = G(t)y$, i.e. $\bar{A}(t) = \left(\begin{array}{cc} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{array} \right)$, relating the Riccati equation (3) with (for $b_0 b_2 \neq 0$, with an integrable one given by*

$$
\frac{dy'}{dt} = D(t)(c_0 + c_1 y' + c_2 y'^2), \tag{7}
$$

where c_i are real numbers, $c_i \in \mathbb{R}$, is that $c_0 c_2 \neq 0$, and:

$$
D^2(t)c_0 c_2 = b_0(t)b_2(t), \qquad \sqrt{\frac{c_0 c_2}{b_0(t)b_2(t)}}\left(b_1(t) + \frac{1}{2}\left(\frac{\dot{b}_2(t)}{b_2(t)} - \frac{\dot{b}_0(t)}{b_0(t)}\right)\right) = c_1.
$$

The unique transformation is then $y' = (b_2(t)c_0)^{1/2}(b_0(t)c_2)^{-1/2} y$.

As a consequence, given (7) if there are constants K, L such that

$$
\sqrt{\frac{L}{b_0(t)b_2(t)}}\left(b_1(t) + \frac{1}{2}\left(\frac{\dot{b}_2(t)}{b_2(t)} - \frac{\dot{b}_0(t)}{b_0(t)}\right)\right) = K
$$

then there exists a time-dependent linear change of variables transforming the given equation into the solvable Riccati equation (7) with $c_1 = K$, $c_0 c_2 = L$ and $D(t)$ is given as above.

The existence of such constant K can be considered a sufficient condition for integrability of the given Riccati equation or the corresponding Milne–Pinney equation.

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